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LETTER TO THE EDITOR

A differential form approach to the equations of self-induced transparency: the prolongation technique

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Abstract. The technique of differential form and exterior calculus has been applied to the analysis of the prolongation structure of the nonlinear equations of self-induced transparency. The process of prolongation yields a completely different scheme of inverse scattering for these equations.

One of the most intriguing features of a nonlinear equation is its soliton-like exact solution, which has some fascinating properties that are worth investigating (Cercignani 1977). The key to such an analysis is the method of inverse scattering transforms (Ablowitz 1977). Until now there has been no logical deduction of the IST scheme for some particular nonlinear equation. Recently Estabrook and Wahlquist (1977a) developed the method of Lie and Cartan, which they successfully implemented for deducing the scheme of IST, Bäcklund transformation for the nonlinear Schrödinger equation and the K–dV equation. Subsequently Morris (1977a–d, 1978) extended the technique to some other cases. Here we analyse the solutions of the nonlinear equations pertaining to the physical phenomenon of self-induced transparency with the help of the above mentioned technique. In this connection it can be mentioned that the first IST framework for the self-induced transparency equations was obtained by Gibbon *et al* (1973) by demanding a complicated pole structure for the A, B, C functions of the inversion scheme of Ablowitz (1977). Here we show that the differential form analysis yields, among other things, a simpler inversion mechanism for the equations under consideration, which could never be obtained by simple guesswork.

Formulation

Consider the equations of self-induced transparency written in the form

$$\partial e/\partial t = s, \qquad \partial r/\partial x = -\mu s, \qquad \partial s/\partial x = eu + \mu r, \qquad \partial u/\partial x = -es \tag{1}$$

in one space and one time dimension. The differential 2-forms which on proper sectioning yield these equations are

$$\begin{aligned} \alpha_1 &= \mathrm{d}e \wedge \mathrm{d}x - s\,\mathrm{d}x \wedge \mathrm{d}t, & \alpha_2 &= \mathrm{d}r \wedge \mathrm{d}t + \mu s\,\mathrm{d}x \wedge \mathrm{d}t, \\ \alpha_3 &= \mathrm{d}u \wedge \mathrm{d}t + es\,\mathrm{d}x \wedge \mathrm{d}t, & \alpha_4 &= \mathrm{d}s \wedge \mathrm{d}t - (eu + \mu r)\,\mathrm{d}x \wedge \mathrm{d}t. \end{aligned} \tag{2}$$

It is rather interesting to observe that these forms under the operation of exterior differentiation belong to the closed ideal generated by them, that is

$$\begin{aligned} d\alpha_1 &= -ds \wedge dx \wedge dt = -dx \wedge \alpha_4 \\ d\alpha_3 &= s\ de \wedge dx \wedge dt + e\ ds \wedge dx \wedge dt = -de \wedge \alpha_1 - e\ dx \wedge \alpha_4 \end{aligned} \tag{3}$$

and so on.

This closed property is the most important one for the search of the Pfaffian system of Estabrook and Wahlquist. So we define the Pfaffians as

$$\omega_k = dy_k + F^k\ dx + G^k\ dt \tag{4}$$

where the y_k's are some prolongation variables, and F^k, G^k are functions depending on the primitive variables e, r, s, u, x and t, along with the y_k's. At this point one should note that in his analysis Morris has classified the forms into two categories—one which defines the variables and the other which yields the nonlinear equations. The former is called the geometric form and the latter the dynamic form. In our case we consider the equation $\partial e/\partial t = s$ to be the defining equation for the variable s, so that α_1 is a geometric form and $\alpha_2, \alpha_3, \alpha_4$ are dynamic forms, because they yield the nonlinear equations when properly sectioned.

Calculation of the Pfaffian system

The computation of the Pfaffian forms begins from the condition that

$$d\omega_k = \sum f_i\alpha_i + \sum \eta_i \wedge \omega_i^k; \tag{5}$$

that is, the exterior derivative $d\omega_k$ will be in the closed ideal generated by the α_i's and ω_k's. Written in full (5) reads

$$\begin{aligned} d\omega_k &= (\partial F^k/\partial\psi^\mu)\ d\psi^\mu \wedge dx + (\partial G^k/\partial\psi^\mu)\ d\psi^\mu \wedge dt \\ &= \sum f_i\alpha_i + (a_1\ dx + b_1\ dt + a_2\ de + a_3\ dr + a_4\ du + a_5\ ds) \wedge (dy^k + F^k\ dx + G^k\ dt) \end{aligned}$$

where ψ^μ is nothing but the collection of all primitive variables. Equating the coefficients of all possible 2-forms we obtain

$$\begin{aligned} &F_r = F_u = F_s = 0,\ G_e = 0, \\ &-sF_e^x + \mu s G_r^k + es G_u^k - (eu + \mu r)G_s^k + F_{y_i}^k G^i - G_{y_i}^k F^i = 0. \end{aligned} \tag{6}$$

The structures of F and G that emerge from the equations obtained by repeated differentiation of (6) are

$$\begin{aligned} G^k &= x_0^k + r x_1^k + u x_2^k + s x_3^k \\ F^k &= x_5^k + e x_4^k. \end{aligned} \tag{7}$$

Substitution of these in the last equation of (6) yields the commutators

$$\begin{aligned} &[x_5, x_0] = 0, \quad [x_4, x_0] = 0, \quad [x_4, x_3] = -x_2, \quad [x_5, x_3] = -x_4 + \mu x_1 \\ &[x_4, x_2] = -x_3, \quad [x_5, x_1] = -\mu x_3, \quad [x_5, x_2] = 0, \quad [x_4, x_1] = 0. \end{aligned} \tag{8}$$

The next important step is the closure of the algebra so obtained by augmenting this set with all its Jacobi identities, which yields

$$\begin{aligned} &[x_1, x_2] = \gamma_1 x_3, \quad [x_1, x_3] = \gamma_1 x_2, \quad [x_4, x_5] = a x_3, \\ &[x_2, x_3] = \lambda x_1 + \sigma x_4, \quad [x_0, x_2] = 0, \quad [x_0, x_3] = 0, \quad [x_0, x_1] = 0. \end{aligned} \tag{9}$$

From the structure of the commutators in (8) and (9) it is clear that x_0 commutes with all of them, so we set

$$x_0 = \sigma \mathbb{1}; \tag{10}$$

that is, some multiple of the unit matrix. Furthermore, x_1, x_2, x_3 and x_4 form a closed Lie algebra, so that one immediately deduces a Casimir-type (Estabrook and Wahlquist 1977b) representation given by

$$\begin{aligned} &x_1 = -\gamma_1(y_2\,\partial/\partial y_3 + y_3\,\partial/\partial y_2), \qquad x_2 = \gamma_1 y_1\,\partial/\partial y_3 - \lambda y_3\,\partial/\partial y_1 - \sigma y_3\,\partial/\partial y_4 - y_4\,\partial/\partial y_3,\\ &x_3 = \gamma_1 y_1\,\partial/\partial y_2 + \lambda y_2\,\partial/\partial y_1 + \sigma y_2\,\partial/\partial y_4 - y_4\,\partial/\partial y_2, \qquad x_4 = y_2\,\partial/\partial y_3 + y_3\,\partial/\partial y_2 \end{aligned} \tag{11}$$

with x_5 given as

$$\begin{aligned} x_5 &= a y_4\,\partial/\partial y_3 + y_3\,\partial/\partial y_4 - \mu y_3\,\partial/\partial y_1 + \mu y_1\,\partial/\partial y_3,\\ x_0 &= \sigma(y_1\,\partial/\partial y_1 + y_2\,\partial/\partial y_2 + y_3\,\partial/\partial y_3 + y_4\,\partial/\partial y_4). \end{aligned} \tag{12}$$

Consistency of these equations demands

$$\lambda/\sigma + \mu = 0, \qquad \gamma_1 = -\lambda.$$

Inverse scattering equation and eigenvalue problem

Substituting these forms of the operators, it is easy to observe that, if we define $Y = (y_1, y_2, y_3, y_4)$ to be a four-component vector, then

$$Y_x = MY, \qquad Y_t = NY \tag{13}$$

where the matrices M and N are given by

$$M = \begin{pmatrix} 0 & 0 & \lambda/\sigma & 0\\ 0 & 0 & e & 0\\ -\lambda/\sigma & e & 0 & 1/\sigma\\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & s\lambda & -\lambda u & 0\\ s\lambda & 0 & -\lambda r & -s\\ u\lambda & -\lambda r & 0 & -u\\ 0 & s\sigma & -\lambda u & 0 \end{pmatrix}. \tag{14}$$

It is rather straightforward to observe that the nonlinear equations result from the consistency of (13) that $Y_{tx} = Y_{xt}$.

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